# Non-Markovian Toy Quantum Chain

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We propose a simple structure for stationary non-Markovian quantum chains in the framework of collisional dynamics of open quantum systems. To this end, we modify the microscopic Markovian system–reservoir model, consider multiple collisions with each of the molecules with an overlap between the collisional time intervals. We show how the equivalent Markovian quantum chain can be constructed with the addition of satellite quantum memory to the system. We distinguish quantum from classical non-Markovianity. Moreover, we define the counts of non-Markovianity by the required number of satellite qubits and bits, respectively. As the particular measure of quantum non-Markovianity the discord of the satellite w.r.t. the system is suggested. Simplest qubit realizations are discussed, and the significance for real system–environment dynamics is also pointed out.

## I.  INTRODUCTION

In recent years we have seen increasing interest in abstract modeling of time evolution of open quantum systems coupled to their environment via different quantum channels. While decrypting fundamental laws is their ultimate goal, inspirations of such models and possible future applications of their results range from solid state physics to quantum biology and quantum information technology [1–4].

Dynamics modeled microscopically as a series of discrete interactions or "collisions" between a central system and environmental (reservoir) *molecules* leads to abstract collisional system-reservoir models. For a memoryless Markovian time evolution the reservoir has the idealized ability of *forgetting* new information faster than the rate of collisions; i.e., after each collision the reservoir would totally relax to its pre-collisional state before the next collision occurs. For the (central) system, this results in what we call Markovian quantum chain. In each step along the chain, the system undergoes decoherence, leakage of information out to the reservoir to be forgotten there. Mathematically, the Markov quantum chain's irreversible dynamics is obtained from reduction of the unitary time evolution of the system-environment compound by means of partial trace over degrees of freedom of the environment. In the special case of stationarity, the mathematical tool of semigroups of completely positive (CP) trace preserving dynamical maps can be used [2–6].

The collisional model of Markovian chains has been studied in numerous works [7, 8]. It will be particularly useful if we want to monitor the system [9–11]. Although we are never allowed to directly measure the system state, we can measure each reservoir molecule after its collision with the system. By their regular selective measurements we can monitor the system state indirectly. The corresponding resolved evolution of the quantum chain is called selective Markovian quantum chain.

A more realistic description requires an account of the reservoir memory and the induced non-Markovianity of the dynamics of the central system (the chain). For analyzing new dynamical properties and quantifying the difference from the simple Markovian processes, several non-Markovianity measures have been proposed so far. These make use of different aspects of non-Markovian (NM) evolutions, e.g., the non-divisibility of the underlying quantum dynamical map [12], the increasing trace distance (i.e.: distinguishability) of two initial quantum states, accompanied by the back-flow of information from the reservoir to the system [13], or the *discord* between system and reservoir [14]. Non-Markovianity is a field of active debates (cf., e.g., [15]) so we emphasize the importance of abstract models to capture fundamental aspects of memory keeping processes.

In this paper we study the structural features of NM open quantum systems by constructing discretized NM processes, i.e., quantum chains. Starting from the generic structure of a Markovian quantum chain we are going to impose a certain NM structure on it. The proposed abstract NM structure corresponds to an open quantum system in a reservoir of non-interacting molecules which only collide with the central system through unitary collisions. The initial, for the moment Markovian, quantum chain can be seen in Fig. 1(a). To engineer the NM element, we allow each molecule to collide with the system more than a single time, say twice, while the molecules' collisional intervals overlap, by assumption, with the collisional periods of both previous and consecutive molecules, cf. Fig. 1(b). Instead of separating what is forgotten and what is not by different timescales, our abstract model has a built-in exact
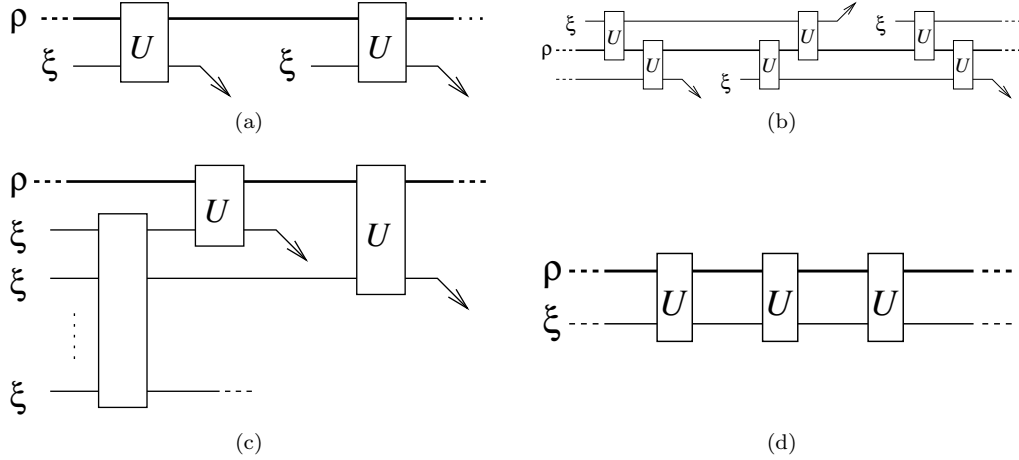
FIG. 1. Quantum chains: different system-reservoir collision models ($\varrho$: system state, $\xi$: reservoir molecule state, $U$ system-molecule unitary collision operator). (a) Standard Markovian chain: the system interacts with each independent molecule once. (b) The proposed NM chain: the system interacts with each independent molecule twice, with overlap between collisional intervals of 'nearest' molecules. (c) NM chain in [18]: the system interacts with each molecule only once, but the molecules are initially entangled. (d) NM chain in [19]: the system interacts with the same molecule multiple times.

memory time. Most importantly, we show how our NM chains become Markovian at the price of attaching *satellite memory* subsystems. Due to our proposed specific NM structure, the identification of the satellite memory within the environment is straightforward compared especially to the efforts requested in oscillatory reservoir models (cf. [16, 17] and Refs. therein). Then a natural *count* of non-Markovianity follows: let it be the size (in qubits) of the requested satellite memory. We can refine this count into informatic *measures* of non-Markovianity, as we show later.

The NM quantum collisional dynamics is not new in itself. Alternative NM structures are shown in Figs. 1(c) and 1(d). Ref. [18] introduces NM mechanism by starting all molecules from an a priori entangled state, cf. Fig. 1(c). This work focuses on the molecular realization of a single NM step instead of general NM chains. In [19], the environment consists of a single molecule and this molecule collides with the system consecutively, many times, cf. Fig. 1(d). An important advantage of our NM structure over these two is that it can invariably host the monitoring of the system, which was so instrumental for quantum Markov chains but becomes complicated in structure Fig. 1(c) and impossible in Fig. 1(d).

In Sec. II the exact notion of a non-selective and a selective quantum Markovian chain is defined. The definition is extended for quantum NM chains in Sec. III. Then Sec. IV contains our new results demonstrated on the toy models of one-qubit NM chains. Based on these results we discuss in Sec. V a new measure of quantum non-Markovianity as well as a distinction between classical and truly quantum non-Markovianity.

## II. QUANTUM MARKOV CHAIN

To define Markov chains, we apply the toolbox containing CP-maps, Kraus matrices, POVMs, selective and non-selective quantum measurements, detailed by the monographs [2–6]. We understand by a non-selective Markov chain a series of quantum states $\varrho_0, \varrho_1, \ldots, \varrho_t \ldots$ of a given system where each state along the chain depends but on the preceeding state respectively:

$$\varrho_{t+1} = \mathcal{M}_t \varrho_t; \qquad t = 0, 1, 2, \ldots, \tag{1}$$

where $\mathcal{M}_t$ are CP maps. If not stated otherwise, we restrict ourselves to stationary chains $\mathcal{M}_t = \mathcal{M}$. A CP map can always be represented by certain Kraus matrices $M_\lambda$ such that

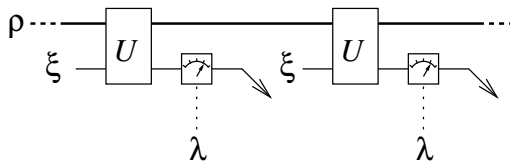$$\mathcal{M}\varrho = \sum_\lambda M_\lambda \varrho M_\lambda^\dagger. \tag{2}$$

FIG. 2. Selective Markov chain: the refinement of the non-selective one in Fig. 1(a). The post-collisional state of the molecule is measured, and yields the random outcome $\lambda$.

Accordingly, non-selective Markov chains (1) can be decomposed into selective Markov chains given by the recursion relation:

$$\varrho_{t+1} = \frac{1}{p_{\lambda,t}} M_\lambda \varrho_t M_\lambda^\dagger. \tag{3}$$

$\lambda$ may represent the random outcome of a POVM measurement characterized by the matrices $M_\lambda$. Here $\mathrm{Tr} M_\lambda \varrho_t M_\lambda^\dagger$ is the outcome probability. As a matter of fact, the selective chain $\varrho_t$ depends on all measurement outcomes $\lambda$ prior to $t$, this dependence is suppressed in our notation. The stochastic average of the selective Markov chain (3) over the measurement outcomes $\lambda$ yields the non-selective Markov chain (1).

We can always construct a microscopic model for a given quantum Markov chain. Consider an abstract ideal gas (reservoir) of identical molecules each in state $\xi$. Independent unitary collisions will generate the CP map $\mathcal{M}$ of (1):

$$\varrho_{t+1} = \mathrm{Tr}_{\mathrm{res}}[U(\varrho_t \otimes \xi)U^\dagger], \tag{4}$$

where $U$ is the collision matrix, $\mathrm{Tr}_{\mathrm{res}}$ is the partial trace over the molecule state. The process is shown in Fig. 1(a). Any map (2) can be generated by a suitable unitary mechanism (4) whereas the choice of $U$ and $\xi$ is never unique [5, 6]. If we inspect, i.e. measure, the post-collisional state of the molecule we get the underlying microscopic model of the selective Markov chain (3), shown in Fig. 2. As to determining Kraus matrices $M_\lambda$, their possible choice is simple if the molecules are prepared in pure state $\xi = |\Psi\rangle \langle\Psi|$:

$$M_\lambda = \langle\lambda| U |\Psi\rangle . \tag{5}$$

We show concrete examples in Sec. IV.

## III.   QUANTUM NON-MARKOV CHAIN

Consider the chain

$$\varrho_t = \mathcal{M}(t)\varrho_0; \quad t = 0, 1, 2, \ldots, \tag{6}$$

where $\mathcal{M}(t)$ is a $t$-dependent CP map. The chain is Markovian, see (1), if $\mathcal{M}(t)$ is *divisible*, i.e. can be written in the following form:

$$\mathcal{M}(t) = \mathcal{M}_t \mathcal{M}(t-1); \qquad t = 1, 2, \ldots, \tag{7}$$

with some sequence of CP maps $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_t, \ldots$. To be clear, we require that

$$\begin{aligned}
\mathcal{M}(1) &= \mathcal{M}_1, \\
\mathcal{M}(2) &= \mathcal{M}_2 \mathcal{M}_1, \\
\mathcal{M}(3) &= \mathcal{M}_3 \mathcal{M}_2 \mathcal{M}_1, \quad \text{e.t.c.}
\end{aligned} \tag{8}$$

If such a sequence $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_t, \ldots$ does not exist, i.e., when $\mathcal{M}(t)$ is not divisible into the same set of factors for all $t$, the chain $\varrho_t$ is NM.

The microscopic mechanism of quantum non-Markovianity can be quite complicated. A recent study [18] considers single qubit maps which are not divisible at all. To model such maps microscopically, a class of finite NM chains has been constructed. Its key mechanism has been independent collisions with entangled molecules, the general structure is shown in Fig. 1(c). Another work [19] considered a single qubit environment, interacting unlimited times with the system qubit via a controlled-rotation. Our approach will be different and elementary. We construct a class of
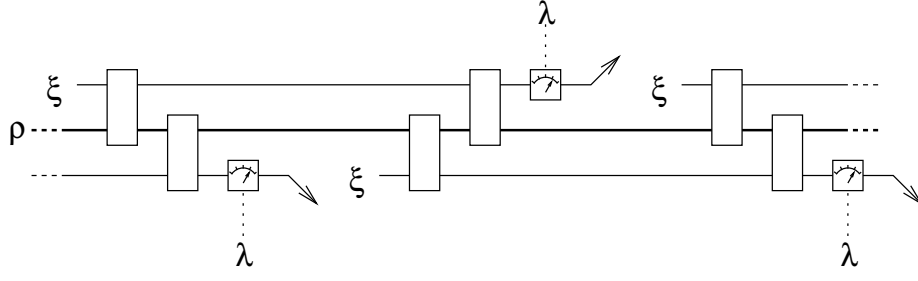
FIG. 3. Selective NM chain: the refinement of the non-selective one in Fig. 1(b). The post-collisional (i.e.: after the double collision) state of the molecule is measured, and yields the random outcome $\lambda$.
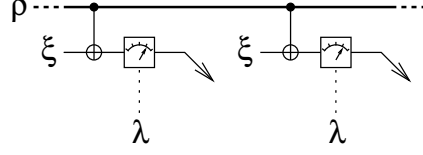


FIG. 4. Selective one-qubit Markov chain: collisional model. The system qubit $\varrho$ interacts (collides) with the reservoir qubits (molecules) $\xi$ via XOR-gates; the post-interaction states of the molecular qubits are measured, yielding the random outcomes $\lambda$.

stationary non-Markovianity. Our molecules are independent, and do not become entangled prior to their collisions with the system. However, they will collide multiple times — twice, for simplicity's sake — and this constitutes the memory mechanism. Fig. 1(b) shows the general NM structure. Unlike the previous NM structure shown in Fig. 1(d), ours allows us, by its construction, to insert measurements without changing the physics of the process. Obviously, we can insert measurements on each molecule after its second collision, as shown in Fig. 3.

Rather than discussing this NM model generally, in what follows we concentrate on its simplest qubit realizations.

## IV. ONE-QUBIT NON-MARKOV CHAINS

In order to highlight the similarities and differences, first, in Sec. IV A, we are going to construct a one-qubit Markov chain, and then, in Secs. IV B and IV C, by extending it, we study two different NM structures.

Throughout this section the molecules constituting the reservoir are single qubits in pure initial states $\xi = |\Psi_\phi\rangle \langle \Psi_\phi|$, where

$$|\Psi_\phi\rangle = \cos\phi \, |0\rangle + \sin\phi \, |1\rangle \,. \tag{9}$$

For later convenience, we write the state of the molecule as

$$|\Psi_\phi\rangle = \exp(i\phi\sigma_y) \, |0\rangle \,. \tag{10}$$

### A. One-qubit Markov chain

Let our central system be a single qubit, and let us construct a Markov chain (1-3), also see Figs. 1(a) and 2. Couple the molecule in state $\xi$ to the system qubit in state $\varrho_t$ via the XOR-gate. The corresponding two-qubit unitary operator $U$ is the following:

$$U = \mathrm{XOR}_{\mathrm{sys-mol}}. \tag{11}$$

The XOR-gate installs the unsharp measurement of the system qubit by the molecular qubit. The strength of the measurement is controlled by the parameter $\phi$ of the molecular pre-collisional state. If $\phi = 0, \pi/2$, the measurement is projective while $\phi \approx \pi/4$ yields a weak measurement [20]; the value $\phi = \pi/4$ decouples the system from the molecules.

The two Kraus matrices are defined by $M_\lambda = \langle\lambda| U |\Psi_\phi\rangle$ for $\lambda = 0, 1$, cf. Eq. (5). Inserting (10), we have

$$M_\lambda = \langle\lambda| U \exp(i\phi\sigma_y) \, |0\rangle \,. \tag{12}$$

To perform the partial trace operation (4) from the four-dimensional state $U(\varrho_t \otimes \xi)U^\dagger$ to the two-dimensional state $\varrho_t$, we use the deeper level selective form (3) for technical reasons.

The unitary evolution of the molecule-system compound $\xi \otimes \varrho_t$ in the product basis $|\text{mol}, \text{sys}\rangle = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ is given by

$$U \exp(i\phi\sigma_y) = \begin{pmatrix} \cos\phi & & -\sin\phi & \\ & \sin\phi & & \cos\phi \\ \sin\phi & & \cos\phi & \\ & \cos\phi & & -\sin\phi \end{pmatrix}. \tag{13}$$

The two Kraus matrices in this representation are simply the left block matrices of $U$:

$$M_0 = \begin{pmatrix} \cos\phi & \\ & \sin\phi \end{pmatrix} \qquad M_1 = \begin{pmatrix} \sin\phi & \\ & \cos\phi \end{pmatrix}. \tag{14}$$

The non-selective Markov chain (1) is as trivial as [?]

$$\varrho_{00}^{t+1} = \varrho_{00}^t \quad \varrho_{11}^{t+1} = \varrho_{11}^t \quad \varrho_{01}^{t+1} = \sin(2\phi)\varrho_{01}^t \tag{15}$$

i.e., the diagonal elements are preserved while the off-diagonals will step towards zero unless we took the singular molecular states with $\phi = \pi/4$:

$$\varrho_\infty = \begin{pmatrix} \varrho_{00}^0 & 0 \\ 0 & \varrho_{11}^0 \end{pmatrix}. \tag{16}$$

We can say that our Markov chain with $\phi \neq \pi/4$ is asymptotically equivalent with a single von Neumann projective measurement. Even if the single collisions are in the weak measurement regime, their cumulative effect is the projective measurement, as is well known, e.g., from [9–11].

## B. Repeated XORs with time overlap

To construct the simplest non-Markov chain, let each molecule interact with the system qubit twice, and let there be an overlap between subsequent collisional periods, see Fig. 5(a). We are interested in stationary chains hence the pattern of double collision will identically repeat itself along the chain. Note, however, that the starting pattern must always be a "broken" one, i.e., in the beginning there is a molecule that collides only once with the central system.

From the classical theory of NM chains with finite memory time we know that adding suitable amount of memory (bits) to the system yields an equivalent Markovian chain. Similarly, we attach a memory qubit to the system qubit. We can make e.g., one distinguished molecule the memory. Consider the resulting scheme in Fig. 5(b), equivalent with Fig. 5(a). The system+memory compound is a two-qubit composite system whose state will be denoted by $\tilde{\varrho}_t$ and, as we see, it undergoes independent collisions with the rest of the molecules of state $\xi$ each. Our trick is that we repeatedly swap all system-environment entanglement into the system-memory compound.

Accordingly, we have obtained a Markovian quantum chain for the system+memory compound $\tilde{\varrho}_t$ instead of the system $\varrho_t$ alone. Markovian structure (1)-(4) applies invariably. The molecule-memory-system compound is a three-qubit system. The three-qubit unitary operation $U$ is the following:

$$U = \text{XOR}_{\text{sys}-\text{mol}} \, \text{SWAP}_{\text{mol}-\text{mem}} \, \text{XOR}_{\text{sys}-\text{mol}}. \tag{17}$$

In the basis $|\text{mol}, \text{mem}, \text{sys}\rangle = \{|000\rangle, |001\rangle, |010\rangle, \ldots |111\rangle\}$, we calculate the collision matrix (17) times $\exp(i\phi\sigma_y)$, with notations $c = \cos\phi$ and $s = \sin\phi$:

$$U \exp(i\phi\sigma_y) = \begin{pmatrix} c & & -s & & & & & \\ & s & & & & & c & \\ s & & c & & & & & \\ & & & c & & & -s & \\ & c & & & & -s & & \\ & s & & & c & & & \\ & & s & & & & c & \\ & c & & & & -s & & \end{pmatrix}. \tag{18}$$
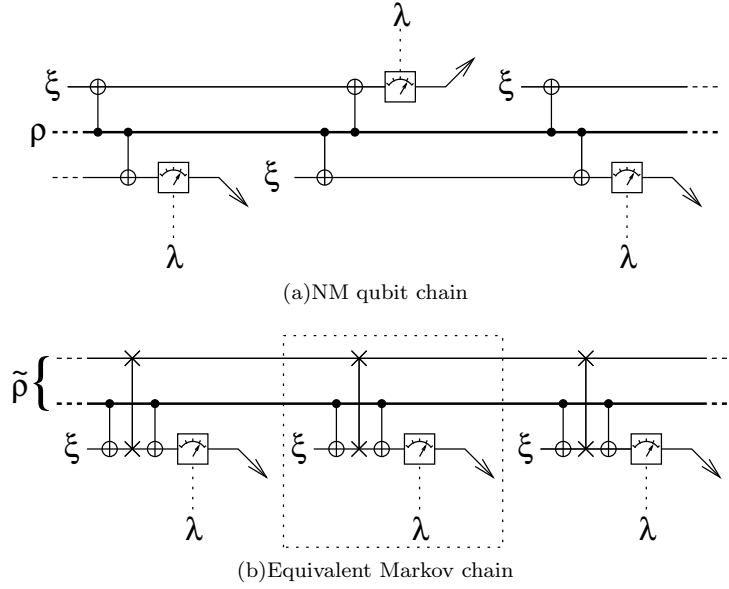
FIG. 5. Equivalent one-qubit NM and two-qubit Markov chains. (a) NM chain: the system qubit $\varrho$ interacts (collides) twice with the same reservoir qubits (molecules) $\xi$ via two XOR-gates, with overlap between 'nearest' molecules; the post-interaction states of the molecular qubits are measured, yielding the random outcomes $\lambda$. (b) Equivalent Markov chain: the system-satellite two-qubit $\tilde{\varrho}$ compound interacts (collides) with the reservoir qubits (molecules) $\xi$ via the swap- and two XOR-gates but without the overlap between collisional intervals of different molecules; the post-interaction states of the molecular qubits are measured, yielding the random outcomes $\lambda$.

According to (12), the two Kraus matrices $M_0$ and $M_1$ in the basis $|\text{mem}, \text{sys}\rangle = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ are given by the upper-left and lower-left $4 \times 4$ blocks, respectively:

$$M_0 = \begin{pmatrix} c & & 0 & \\ & 0 & & s \\ s & & 0 & \\ & 0 & & c \end{pmatrix} \qquad M_1 = \begin{pmatrix} 0 & & c & \\ & s & & 0 \\ 0 & & s & \\ & c & & 0 \end{pmatrix}. \tag{19}$$

Using these Kraus matrices, the non-selective Markov chain (1) reads:

$$\tilde{\varrho}_{t+1} = \begin{pmatrix} c^2(\tilde{\varrho}_{00}^t + \tilde{\varrho}_{22}^t) & cs(\tilde{\varrho}_{03}^t + \tilde{\varrho}_{21}^t) & cs(\tilde{\varrho}_{00}^t + \tilde{\varrho}_{22}^t) & c^2(\tilde{\varrho}_{03}^t + \tilde{\varrho}_{21}^t) \\ cs(\tilde{\varrho}_{30}^t + \tilde{\varrho}_{12}^t) & s^2(\tilde{\varrho}_{11}^t + \tilde{\varrho}_{33}^t) & s^2(\tilde{\varrho}_{30}^t + \tilde{\varrho}_{12}^t) & cs(\tilde{\varrho}_{11}^t + \tilde{\varrho}_{33}^t) \\ cs(\tilde{\varrho}_{00}^t + \tilde{\varrho}_{22}^t) & s^2(\tilde{\varrho}_{03}^t + \tilde{\varrho}_{21}^t) & s^2(\tilde{\varrho}_{00}^t + \tilde{\varrho}_{22}^t) & cs(\tilde{\varrho}_{03}^t + \tilde{\varrho}_{21}^t) \\ c^2(\tilde{\varrho}_{30}^t + \tilde{\varrho}_{12}^t) & cs(\tilde{\varrho}_{11}^t + \tilde{\varrho}_{33}^t) & cs(\tilde{\varrho}_{30}^t + \tilde{\varrho}_{12}^t) & c^2(\tilde{\varrho}_{11}^t + \tilde{\varrho}_{33}^t) \end{pmatrix}. \tag{20}$$

On the r.h.s. we can identify the diagonal part of the system density matrix $\varrho_{00}^t = (\tilde{\varrho}_{00}^t + \tilde{\varrho}_{22}^t)$ and $\varrho_{11}^t = (\tilde{\varrho}_{00}^t + \tilde{\varrho}_{33}^t)$ as well as the specific correlation $C_{x-}^t = \text{Tr}(\sigma_x \otimes \sigma_- \tilde{\varrho}_t) = (\tilde{\varrho}_{03}^t + \tilde{\varrho}_{21}^t)$ between the memory and the system. Observe that all these quantities are invariants along our NM chain. Not surprisingly then, the chain will immediately reach the stationary state $\tilde{\varrho}_1 = \tilde{\varrho}_2 \cdots = \tilde{\varrho}_\infty$ fully parametrized by the above invariants:

$$\tilde{\varrho}_\infty = \begin{pmatrix} c^2 \varrho_{00}^0 & cs C_{x-}^0 & cs \varrho_{00}^0 & c^2 C_{x-}^0 \\ cs C_{x+}^0 & s^2 \varrho_{11}^0 & s^2 C_{x+}^0 & cs \varrho_{11}^0 \\ cs \varrho_{00}^0 & s^2 C_{x-}^0 & s^2 \varrho_{00}^0 & cs C_{x-}^0 \\ c^2 C_{x+}^0 & cs \varrho_{11}^0 & cs C_{x+}^0 & c^2 \varrho_{11}^0 \end{pmatrix}. \tag{21}$$

The free parameter $C_{x\pm}^0$ depends on the starting "broken" pattern. In the case shown on Figs. 5(a) and 5(b), it can be set to zero, assuming uncorrelated initial satellite memory with $\langle \sigma_x \rangle = 0$. The above stationary state of the memory+system compound then becomes a separable, disentangled state:

$$\tilde{\varrho}_\infty = \varrho_{00}^0 |\Psi_\phi\rangle \langle \Psi_\phi| \otimes |0\rangle \langle 0| + \varrho_{11}^0 |\Psi_\phi'\rangle \langle \Psi_\phi'| \otimes |1\rangle \langle 1|, \tag{22}$$

where the memory state $|\Psi'_\phi\rangle = s\,|0\rangle + c\,|1\rangle$ is not orthogonal to $|\Psi_\phi\rangle = c\,|0\rangle + s\,|1\rangle$ (unless $\phi = \pi/4$). The stationary density matrix $\varrho_1 = \varrho_2 \cdots = \varrho_\infty$ of our central system is trivial:

$$\varrho_\infty = \mathrm{Tr}_{\mathrm{mem}}\tilde{\varrho}_\infty = \begin{pmatrix} \varrho_{00}^0 & 0 \\ 0 & \varrho_{11}^0 \end{pmatrix}. \tag{23}$$

It is the mixture of the basis states with the initial population, as if $\varrho_0$ were ideally measured by the collective of the molecules, like in case of the Markovian chain, cf. Sec. IV A.

The repeated-XOR model has trivial invariants, this is the reason it reaches the asymptotic state in one step already.

## C.   Distributed XORs with time overlap

In order to make a NM chain evolve asymptotically towards the stationary state we must assure that there are fewer trivial invariants. Let us go back for a moment to the Markovian chain with single XOR collisions (13) and introduce a NM variant. Suppose that each XOR-collision takes a finite time and subsequent XOR operations overlap in time. To construct a simplest discrete model of time-distributed XOR, we make XOR in two steps separated by unit time. Let both steps correspond to the following square-root of XOR:

$$\sqrt{\mathrm{XOR}} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \sqrt{i/2} & -i\sqrt{i/2} \\ & & -i\sqrt{i/2} & \sqrt{i/2} \end{pmatrix}. \tag{24}$$

(Here the convention $\sqrt{i/2} = e^{i\pi/8}/\sqrt{2}$ has been chosen.) Our non-Markov model is shown in Fig. 5(a), with the XOR collisions replaced by the above $\sqrt{\mathrm{XOR}}$.

Obviously, we have the equivalent Markov chain, like in Fig. 5(b), and we can perform the same calculations as before. We calculate the unitary matrix

$$U = \sqrt{\mathrm{XOR}}_{\mathrm{sys-mol}}\, \mathrm{SWAP}_{\mathrm{mol-mem}}\, \sqrt{\mathrm{XOR}}_{\mathrm{sys-mol}} \tag{25}$$

times $\exp(i\phi\sigma_y)$, and read out the two Kraus matrices:

$$M_0 = \begin{pmatrix} c & 0 \\ i\bar{\beta} & \bar{\beta} \\ s & 0 \\ \beta & -i\beta \end{pmatrix} \qquad M_1 = \begin{pmatrix} 0 & c \\ & \bar{\beta} & i\bar{\beta} \\ 0 & & s \\ & -i\beta & \beta \end{pmatrix}, \tag{26}$$

with $\beta = e^{i\phi}/2$. These Kraus matrices in (2) yield, after direct calculations, the following Markovian chain (1) for the memory-system compound:

$$\tilde{\varrho}_{t+1} = \begin{pmatrix} c^2(\tilde{\varrho}_{00}^t + \tilde{\varrho}_{22}^t) & c\beta\Delta_t & cs(\tilde{\varrho}_{00}^t + \tilde{\varrho}_{22}^t) & ic\bar{\beta}\Delta_t \\ c\beta\bar{\Delta}_t & 1/2(\tilde{\varrho}_{11}^t + \tilde{\varrho}_{33}^t) & s\beta\bar{\Delta}_t & 2i\bar{\beta}^2(\tilde{\varrho}_{11}^t + \tilde{\varrho}_{33}^t) \\ cs(\tilde{\varrho}_{00}^t + \tilde{\varrho}_{22}^t) & s\beta\Delta_t & s^2(\tilde{\varrho}_{00}^t + \tilde{\varrho}_{22}^t) & is\bar{\beta}\Delta_t \\ -c\beta i\bar{\Delta}_t & -2i\beta^2(\tilde{\varrho}_{11}^t + \tilde{\varrho}_{33}^t) & -s\beta i\bar{\Delta}_t & 1/2(\tilde{\varrho}_{11}^t + \tilde{\varrho}_{33}^t) \end{pmatrix}, \tag{27}$$

where

$$\Delta_t = -i(\tilde{\varrho}_{01}^t + \tilde{\varrho}_{23}^t) + (\tilde{\varrho}_{03}^t + \tilde{\varrho}_{21}^t) \tag{28}$$

and it satisfies a closed evolution equation:

$$\Delta_{t+1} = \sin(2\phi)\Delta_t, \tag{29}$$

therefore this is a convenient parametrization. It follows from (27) that the matrix elements not containing $\Delta$ will take their final stationary values immediately after the first step, just like in our previous model in Sec. IV B. However, the elements with $\Delta$ show an exponential relaxation (29) toward zero if $\sin(2\phi) \neq 0$. The relaxation of $\Delta_t$ governs the asymptotic diagonalization of the system density matrix:

$$\varrho_t = \mathrm{Tr}_{\mathrm{mem}}\tilde{\varrho} = \begin{pmatrix} \varrho_{00}^{t=0} & (1 + i\sin(2\phi))\Delta_t/2 \\ (1 - i\sin(2\phi))\bar{\Delta}_t/2 & \varrho_{11}^{t=0} \end{pmatrix}. \tag{30}$$

Indeed, for $t = \infty$ one has $\Delta_\infty = 0$, the system density matrix $\varrho_\infty$ becomes diagonal with the initial populations $\varrho_{00}^0, \varrho_{11}^0$. As to the stationary density matrix $\tilde{\varrho}_\infty$ of the memory-system compound, the Markov chain (27) yields a separable state again, like in our previous model in Sec. IV B. This time we get

$$\tilde{\varrho}_\infty = \varrho_{00}^0 |\Psi_\phi\rangle \langle \Psi_\phi| \otimes |0\rangle \langle 0| + \varrho_{11}^0 |\Psi_\phi'\rangle \langle \Psi_\phi'| \otimes |1\rangle \langle 1|, \tag{31}$$

where the memory state $|\Psi_\phi'\rangle = (|0\rangle + ie^{2i\phi} |1\rangle)/\sqrt{2}$, different from $|\Psi_\phi'\rangle$ in the model in Sec. IV B, is never orthogonal to $|\Psi_\phi\rangle$. This is what makes our system *quantum* NM, a distinction discussed in the next section.

## V. CLASSICAL/QUANTUM NON-MARKOVIANITY

Our construction of NM quantum chains may become universal if we allow for more than two repeated collisions with a single molecule and/or overlaps of collision periods between more than two molecules. Whether or not an arbitrary NM chain will be reducible to ours remains an open theoretical issue. Our class of NM chains is unique for at least one thing: we can always identify the satellite memory to make the time evolution of the resulting system-memory compound a Markovian chain. We can always read-out from the circuit of the NM chain how many qubits are needed for the satellite memory. This number of qubits is a useful *count* of non-Markovianity of our chain, e.g., in computational simulation this number gives an upper bound on how much data should be stored dynamically together with the data of our system of interest.

However, this count may be significantly larger than the amount of information to be contained in the satellite. Suppose that we have determined the minimum number of qubits needed for the satellite, so the count of non-Markovianity is known. Then a suitable informatic quantity, like the mutual information [5, 6] of the satellite memory (M) and the system (S) might play the role of non-Markovianity *measure* $\mu_{NM}$. In the stationary regime one gets

$$\mu_{NM} = I(S : M) = H(S) + H(M) - H(S, M) \tag{32}$$

where $H$ stands in turn for the von Neumann entropy of the S-state $\mathrm{Tr}_{\mathrm{mem}} \tilde{\varrho}_\infty = \varrho_\infty$, of the M-state $\mathrm{Tr}_{\mathrm{sys}} \tilde{\varrho}_\infty$, and of the composite state $\tilde{\varrho}_\infty$.

We should stop for a second, and distinguish *quantum* from *classical* non-Markovianity. The attentive reader may have noticed that our construction of quantum NM chains in itself has nothing particular for quantum chains, it roots in a similar construction of classical NM chains. In fact, any classical NM chain can be represented by an equivalent Markovian chain if we assign a sufficient satellite memory. The minimum size of the satellite memory (e.g.: in bits) is the natural count of non-Markovianity, the mutual information may be the measure of non-Markovianity. In the case of a generic quantum NM chain, it may happen that the minimum satellite memory still consists of bits, qubits are not required at all. In this case we say that the chain is *classically NM*, and its quantum non-Markovianity is zero. It is remarkable that both NM chains in Sec. IV have turned out to be quantum NM. If we look at the composite states (22,31) of the system+satellite compound, we see that classical satellite bits would not work, we need satellite qubits. Although we got zero stationary entanglement between the system and satellite, it does not mean the lack of quantumness. The price of getting rid of quantum non-Markovianity cannot be paid in classical bit instead of qubit. The count of quantum non-Markovianity is 1 for both NM chains. In order to distinguish quantum from classical non-Markovianity we use the notion of discord [21]. The classical measure of non-Markovianity may be defined by

$$\mu_{NM}^{cl} = J(S : M)_{\{\Pi_j^M\}} = H(S) - H(S|\{\Pi_j^M\}) \tag{33}$$

where the rightmost term means the average von Neumann entropy of S when the satellite memory is undergoing the projective measurement via the set $\Pi_j^M$. The quantum non-Markovianity measure is the discord itself:

$$\mu_{NM}^{qu} = I(S : M) - J(S : M)_{\{\Pi_j^M\}}. \tag{34}$$

The sum of quantum and classical non-Markovianity measures yields the total measure $\mu_{NM}$.

We can check our proposal on the two models of Sec. IV. Since the memory states $|\Psi_\phi\rangle, |\Psi_\phi'\rangle$ are non-orthogonal, we get non-zero discord. If they were orthogonal, we could get zero discord, and also we could measure the memory after each collision so that a single classical bit $0, 1$ could be retained instead of the memory qubit: the chain would be classically NM in the stationary regime, with non-Markovianity count 1 (bit).
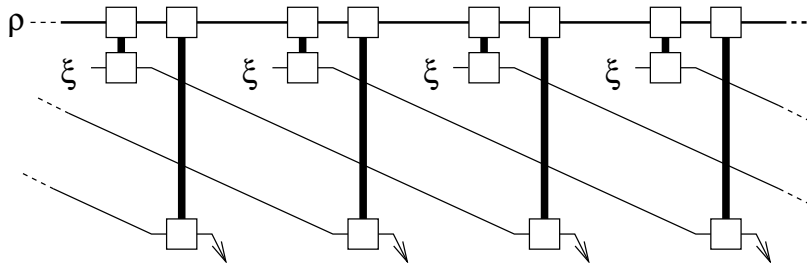
FIG. 6. NM quantum chain of advanced overlap: the system interacts with each independent molecule twice, with overlap between collisional intervals of 'nearest' and 'next-to-nearest' molecules. (Boxes connected with vertical lines represent the bipartite unitary collisions.)

## VI. BRIEF SUMMARY AND OUTLOOK

In the framework of the abstract collision model of system-reservoir interactions, we constructed quantum chains meeting the minimal requirements for NM stationary time evolution. A method of systematic construction of the equivalent Markovian quantum chains using explicit memory allocation is discussed. Due to the transparent NM structure, we can always identify a well-defined part of the reservoir as the memory, this part is called the satellite memory of the system. The time evolution of the system-plus-satellite-memory compound is Markovian. We suggest a novel distinction: a given quantum NM chain is either quantum NM (if the satellite requires qubits) or classical NM (if bits suffice for the satellite). Accordingly, we suggest the numbers of satellite qubits and bits, respectively, as counts of quantum/classical non-Markovianity of a given quantum chain. The mutual information and discord is proposed to measure non-Markovianity and quantum non-Markovianity, respectively.

Two examples of one-qubit NM chains are discussed. In both examples, the corresponding satellite memory is a single qubit. The stationary state is exactly calculable, the non-vanishing discord of the satellite qubit w.r.t. the system qubit indicates that our examples are quantum non-Markovian.

Although our calculations are restricted for non-selective NM chains, the structural transparency allows for the selective refinements. Option of monitoring the NM system is inherent in the model.

For achieving a higher level non-Markovianity than discussed in our work (Fig. 1(b)), one can construct a similar chain with more collisions per molecule and/or longer intervals of overlap. In Fig. 6, e.g., the number of collisions is invariably two whereas the lengths of overlapping intervals have been increased for three collisions. In return, the equivalent Markovian structure needs two satellite qubits instead of one, the non-Markovianity count is 2. The ability to handle complex NM processes may pave the way to the construction of environmental interaction models which better approximate reality.

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